Uncertainty Analysis 1

1 Estimating uncertainty of a measurement

There is an uncertainty involved in any measurement made in experimental work. Any measurement device or procedure has an inherent limit in precision. For example, with a standard ruler, you can usually measure a distance to the nearest millimeter. It's important in physics, when reporting a numerical result, to provide an estimate of the uncertainty. This provides **essential** information: the degree of confidence in the precision of the measurement. There are two basic types of uncertainty we will deal with: uncertainty from the the precision of the measuring device, and uncertainty from random error.

1.1 Precision of the measuring device

When you use a device with a scale, such as a ruler, or graduated cylinder, or a speedometer, the precision of the scale is indicated by the distance between closest tick marks. With digital devices, like a stopwatch, the precision is indicated by the smallest decimal place. With any of these measuring devices, we usually estimate the associated uncertainty as half the smallest division. For a ruler with millimeter tick marks, then, the uncertainty in a length measurement made with that ruler would be $\pm$ 0.5 mm.

1.2 Random error and repeated measurements

Often, however, the uncertainty in a measurement is not limited by the precision of measuring device, but by the procedure for measurement. Consider a researcher measuring the falling time of an object, using a stopwatch with 1 millisecond precision. He repeats the measurement several times and collects the following data:

0.452 s, 0.488 s, 0.511 s, 0.443 s, 0.501 s, 0.526 s, 0.492 s, 0.550 s, 0.469 s, 0.471 s

The researcher is measuring the same event, which presumably has a definite time duration, but there is greater variation in the measurement than the precision of the stopwatch (1 ms). This is likely due to the finite reaction time of the researcher, which results in a random variation of the time values.

This type of random variation occurs in all types of measurements and is called **random error**. When we measure something multiple times, we try to control the process so that it occurs in the same way every time. However, some things are outside our control, like the reaction time when using a stopwatch. **Random error often increases the uncertainty above the precision of the measuring device!**

The best way to determine if there is random error that is larger than the precision of your measuring device is to perform the measurement several times.

1.3 Using statistics for repeated measurements

A good way to minimize the uncertainty in a measurement is to repeat it many times. Statistics is the branch of mathematics that allows you to calculate uncertainty in this case.

Mean:

The mean is the average value of a set of numbers, and is what you will use for the best estimate of your measurement. For a set of N measured values, the mean is given by:

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_N}{N} = \frac{1}{N}\sum_{i=1}^{N} x_i \quad (1)$$

Exercise 1 Using your calculator or Excel, find the mean of the time measurements listed above. Does this seem like a reasonable representative value for these 10 numbers?

Gaussian distribution::

When most measurements are repeated a very large number of times we expect that half of the data will be $\geq$ the average and half will be $\leq$ the average. We also expect that most of the results will be fairly close to the average and a smaller number of results will be further from the average. When plotted in a histogram, data with random error will form a bell-shaped curve, known as a normal or Gaussian distribution. An example of a Gaussian distribution is shown in the figure below.

Standard deviation:

The uncertainty in a large number of measurements with random error is defined by the standard deviation, denoted as σ. The formal mathematical definition of the standard deviation is:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \overline{x})^2}{N-1}} \quad (2)$$

where $\overline{x}$ (given in equation 1) is the mean and N is the number of measurements (x_i). However, in lab you will most often use Excel or other mathematical software to do the calculation for you.

An intuitive way to understand the standard deviation is that about 2/3 of the measurements will have a value that falls within the range: ($Average \pm\sigma$), about 95% of the data fall within the range: ($Average \pm 2\sigma$), and more than 99% of the data fall within the range: ($Average \pm 3\sigma$).

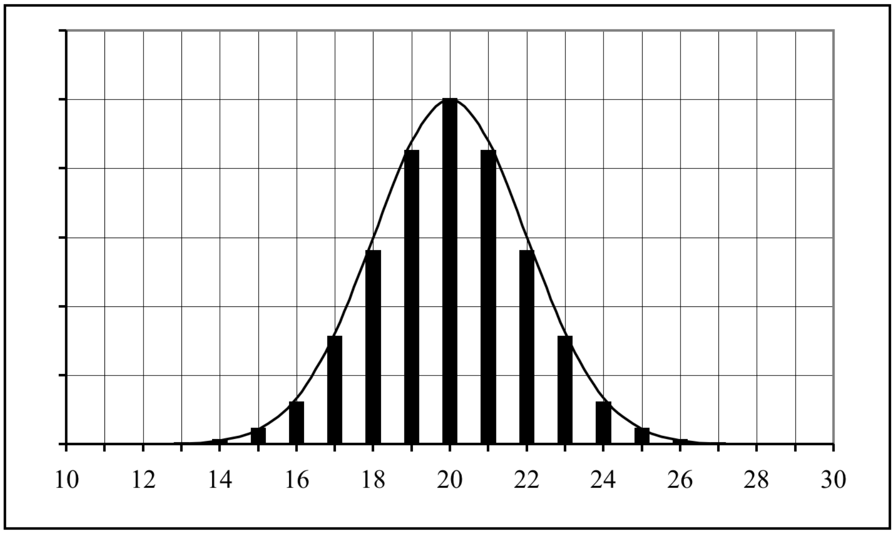

Figure 1: A histogram of measurements for a normal distribution with an average of 20 and a standard deviation of 2. The vertical axis shows the number of occurrences of the value on the horizontal axis.

Exercise 2 Using your calculator or Excel, find the standard deviation of the time measurements listed above. Calculate the 1σ range of values, and determine how many of the measurements fall with in this range. Do the same for 2σ and 3σ ranges.

2 Reporting uncertainty

2.1 Standard Form

You should always report uncertainty for any measurement. There is a **standard form** for reporting uncertainty:

$$(x \pm \delta x) units \tag{3}$$

In this statement, x is the measured value and δx is its uncertainty. For example, if you measure the diameter of a marble to be $15.2mm$ with an uncertainty of $0.5mm$, you would express this in standard form as:

$$(15.2 \pm 0.5) mm \tag{4}$$

Notice that both the measurement and the uncertainty have units! This standard form actually expresses a **range** of length values. The diameter of the marble falls somewhere within the range of $14.9mm$ to $15.7mm$.

2.2 Rounding and Significant figures

In physics, we follow two simple rules for rounding:

1. Round the uncertainty to **ONE** significant figure.

2. Round the measured value to the **same decimal place** as the uncertainty.

For example, if you measured the size of an object, and found the following values for mean and standard deviation:

Mean: $L = 0.54682m$
St Dev: $\delta L = 0.038216m$

You would round as follows: First, the standard deviation to one sig fig: $\delta L = 0.04m$, and second, the mean to the same decimal place: $L = 0.55m$. In standard form:

$L = (0.55 \pm 0.04)m$

Exercise 3 Report the time measurement for the values above in standard form, using your calculated values for the mean and standard deviation. Be sure to round correctly!