Uncertainty Analysis 2

1 Propagating uncertainty

In many experiments, you will use measured quantities in calculations to determine a value of interest. For example, if you wanted to calculate the volume of a marble, you might use calipers to measure its diameter, and then calculate the volume using the following equation:

$$V = \frac{4\pi}{3}\left(\frac{d}{2}\right)^3.$$

The measurement of diameter will of course have some associated uncertainty. The uncertainty in diameter will then influence the uncertainty in the volume, but how, exactly? To determine this, we must **propagate the uncertainty** in the calculation. This handout will outline the basic steps for propagating uncertainty.

1.1 Simplified Propagation Rules

First, below we list the rules for propagating uncertainty through for different basic mathematical procedures. Then we will show some examples of how to apply these rules.

Assume two measured quantities: $A \pm \delta A$ and $B \pm \delta B$. The rules are:

1. **Addition**
   If C = A + B or C = A - B, then
   $$\delta C = \delta A + \delta B \tag{1}$$

2. **Multiplication or Division**
   If C = A×B or C = A/B, then
   $$\frac{\delta C}{C} = \frac{\delta A}{A} + \frac{\delta B}{B} \tag{2}$$

3. **Multiplication or Division by a Constant**
   If C = (*const*)×A (where *const* is a number without uncertainty), then
   $$\frac{\delta C}{C} = \frac{\delta A}{A} \tag{3}$$

4. **Raising to a Constant Power**
   If C = $A^{|q|}$ (where q is a number without uncertainty), then

$$\frac{\delta C}{C} = |q| \times \frac{\delta A}{A} \tag{4}$$

1.2 Example:

What is the volume of a marble and its uncertainty, if the measured diameter is 15.3 mm ± 0.8 mm?
Solution:
First, find the volume using:

$$V = \frac{4\pi}{3}\left(\frac{d}{2}\right)^3.$$

The volume is V = $(4\pi/3)$*$(15.3/2)^3$ mm^3 = 1912.32 mm^3. Do not round this value yet.

To calculate the uncertainty, follow each calculation performed on the diameter value, propagating the uncertainty at each step. First, the diameter is divided by the constant value 2 to find the radius. Then by rule 3:

$$\frac{\delta r}{r} = \frac{\delta d}{d}$$

Next, the radius is raised to the 3rd power, so by rule 4:

$$\frac{\delta r^3}{r^3} = 3\frac{\delta r}{r}.$$

Then this quantity is multiplied by the constant $(4\pi/3)$, so we apply rule 3 again to find:

$$\frac{\delta V}{V} = \frac{\delta r^3}{r^3}$$

Combining this with the previous equations gives:

$$\frac{\delta V}{V} = 3\frac{\delta d}{d}.$$

Finally, we can multiply both sides of the equation by V and plug in values.

$$\delta V = 3V\frac{\delta r}{r} = 3(1912.32\,mm^3)\frac{15.3\,mm}{0.8\,mm} = 298.3\,mm^3.$$

In standard form, the volume with uncertainty is (1.9 ± 0.3) $\times 10^3$ mm^3.

BE CAREFUL not to round all the numbers before you propagate the uncertainty. It is okay to have too many significant figures during intermediate steps in your calculation. Only a value that will be reported as collected data or a final result should be rounded. If you round your values too early, you will introduce mathematical errors in your value.

1.3 Exercises:

Propagate the uncertainty for each of the following calculated values.

1. Find A and δA for the following equation and measured values:

$$A = A_0 + XY$$

where $A_0 = (25.4 \pm 0.5)cm^2$, $X = (3.6 \pm 0.1)cm$, and $Y = (7.0 \pm 0.1)cm$

2. Find x and δx for the following equation and measured values:

$$x = \frac{1}{2}at^2$$

where $a = (0.71 \pm 0.01)m/s^2$ and $t = (1.2 \pm 0.3)s$